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NONLINEAR EQUATIONS OF ELASTIC DEFORMATION OF PLATES

A. E. Alekseev

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A method for constructing nonlinear equations of elastic deformation of plates with boundary conditions for stresses and displacements at the face surfaces in an arbitrary coordinate system is proposed. The initial three-dimensional problem of the nonlinear theory of elasticity is reduced to a one-parameter sequence of two-dimensional problems by approximating the unknown functions by truncated series in Legendre polynomials. The same unknowns are approximated by different truncated series. In each approximation, a linearized system of equations whose differential order does not depend on the boundary conditions at the face surfaces which can be formulated in terms of stresses or displacements is obtained.

Generally, the nonlinear equations governing the elastic deformation of plates are three-dimensional equations of the nonlinear theory of elasticity. The dimensionality of the initial problem can be reduced by various methods. The effective methods of reducing a three-dimensional problem to a two-dimensional problem are based on the expansion of the desired quantities in series in terms of Legendre polynomials (see, e.g., [1]). Ivanov [2] proposed a method for reducing the dimensionality of the linear problems of elastic deformation of the plates and shells of constant thickness with arbitrary boundary conditions for displacements and stresses at the face surfaces that is based on several approximations of the same unknown functions by truncated series in Legendre polynomials. Using this method, Alekseev [3] obtained a one-parameter family of successive approximations of the equations of a deformable layer of variable thickness in an arbitrary coordinate system. In the present paper, the method proposed in [2, 3] is generalized to the nonlinear deformation of elastic plates.

1. Equations of the Nonlinear Theory of Elasticity in an Arbitrary Curvilinear System of Coordinates. We consider an arbitrary curvilinear system of Lagrangian coordinates ξ^i ($i = 1,\ 2,\ 3$). The equations of equilibrium for a continuous medium are written in the vector form

$$\hat{\boldsymbol{t}}^i_{,i} + \hat{\boldsymbol{f}} = 0, \qquad \hat{\boldsymbol{t}}^i = J\boldsymbol{t}^i, \qquad \hat{\boldsymbol{f}} = J\boldsymbol{f}, \qquad \boldsymbol{t}^i = \sigma^{ij}\boldsymbol{g}_j; \tag{1.1}$$

$$\boldsymbol{g}_i \times \hat{\boldsymbol{t}}^i = 0, \qquad J = \boldsymbol{g}_1 \cdot (\boldsymbol{g}_2 \times \boldsymbol{g}_3). \tag{1.2}$$

Here $\boldsymbol{g}_i$ is the covariant basis of the curvilinear system of coordinates ξ^i in a deformed state, $J = \boldsymbol{g}_1 \cdot (\boldsymbol{g}_2 \times \boldsymbol{g}_3)$ is the Jacobian of the transformation of coordinates, σ^{ij} are the components of the Cauchy stress tensor, and $\boldsymbol{f}$ is the vector of body forces. Equality (1.2) is implied by the symmetry of the stress tensor.

The components of the Green–Lagrange strain tensor ε_{ij} are related to the displacement vector $\boldsymbol{u}$ by the nonlinear relations

$$2\varepsilon_{ij} = \overset{0}{\boldsymbol{g}}_i \cdot \boldsymbol{u}_{,j} + \overset{0}{\boldsymbol{g}}_j \cdot \boldsymbol{u}_{,i} + \boldsymbol{u}_{,i} \cdot \boldsymbol{u}_{,j}, \tag{1.3}$$

where $\overset{0}{\boldsymbol{g}}_i$ is the covariant basis of the coordinate system ξ^i in the undeformed state, and the zero above the symbol shows that the quantity corresponds to the undeformed state.

The covariant basis vectors of the coordinate system ξ^i in the deformed state have the form

$$\boldsymbol{g}_i = \overset{0}{\boldsymbol{g}}_i + \boldsymbol{u}_{,i}. \tag{1.4}$$

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090.
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With allowance for (1.3) and (1.4), the strain variations $\delta\varepsilon_{ij}$ are given by

$$2\delta\varepsilon_{ij} = \boldsymbol{g}_i \cdot \delta\boldsymbol{u}_{,j} + \boldsymbol{g}_j \cdot \delta\boldsymbol{u}_{,i}. \tag{1.5}$$

Hooke's law is taken in the form

$$\tau^{ij} = \overset{0}{C}{}^{ijks}\varepsilon_{ks}, \tag{1.6}$$

where τ^{ij} are the covariant components of the second Piola–Kirchhoff stress tensor, $\overset{0}{C}{}^{ijks}$ are the contravariant components of the fourth-rank tensor which are subjected to the symmetry conditions $\overset{0}{C}{}^{ijks} = \overset{0}{C}{}^{jiks} = \overset{0}{C}{}^{ksij}$.

In the coordinate system ξ^i, the equality

$$\overset{0}{J}\tau^{ij} = J\sigma^{ij} \tag{1.7}$$

is valid. Here $\overset{0}{J} = \overset{0}{\boldsymbol{g}}_1 \cdot (\overset{0}{\boldsymbol{g}}_2 \times \overset{0}{\boldsymbol{g}}_3)$ is the Jacobian of the transformation of coordinates in the undeformed-state metric.

Below, the boundary conditions refer to the undeformed state.

We assume that the boundary of an undeformed body $\overset{0}{S}$ consists of two parts: the part $\overset{0}{S}_u$, where the displacements

$$\boldsymbol{u}\Big|_{\overset{0}{S}_u} = \boldsymbol{u}_* \tag{1.8}$$

are specified, and the part $\overset{0}{S}_\sigma$, where the stresses

$$\tau^{ij}\boldsymbol{g}_j\overset{0}{\nu}_i\Big|_{\overset{0}{S}_\sigma} = \boldsymbol{p}_*. \tag{1.9}$$

are specified. Here $\overset{0}{\nu}_i = \overset{0}{\boldsymbol{\nu}} \cdot \overset{0}{\boldsymbol{g}}_i$ ($\overset{0}{\boldsymbol{\nu}}$ is the outward normal to the boundary $\overset{0}{S}$), $\boldsymbol{u}_*$ and $\boldsymbol{p}_*$ are vector functions specified on $\overset{0}{S}$.

Given the equations of equilibrium (1.1) and (1.2) and the boundary conditions (1.8) and (1.9), we can write the virtual-work principle

$$\int\limits_{\overset{0}{V}} \tau^{ij}\delta\varepsilon_{ij}\, d\overset{0}{V} = \int\limits_{\overset{0}{S}_\sigma} \boldsymbol{p}_* \cdot \delta\boldsymbol{u}\, d\overset{0}{S}\,. \tag{1.10}$$

Relative to the undeformed state, the boundary-value problem (1.1)–(1.9) is assumed to be the initial boundary-value problem of the nonlinear theory of elasticity.

2. Expansion of Functions in Terms of Legendre Polynomials. We consider a plate of constant thickness $2h$. In the undeformed state, the plate occupies the volume $\overset{0}{V}$ bounded by the face surfaces $\overset{0}{S}{}^+$ and $\overset{0}{S}{}^-$ and edge surface $\overset{0}{\Sigma}$.

Let x_i be the Cartesian coordinates. In the undeformed state, the middle surface of the plate coincides with the coordinate plane $x_3 = 0$, and the face surfaces $\overset{0}{S}{}^+$ and $\overset{0}{S}{}^-$ correspond to $x_3 = +h$ and $x_3 = -h$, respectively.

We choose the curvilinear system of Lagrangian coordinates ξ^k in such a manner that the coordinate axis ξ^3 coincides with the x_3 axis in the undeformed state. The coordinates x_3 and ξ^3 are related by the formula $x_3 = h\xi^3$. In the undeformed state, the position of any internal point in a plate of volume $\overset{0}{V}$ is given by the vector function of curvilinear coordinates ξ^k

$$\overset{0}{\boldsymbol{R}}(\xi^k) = \overset{0}{\boldsymbol{r}}(\xi^\alpha) + h\boldsymbol{n}\xi^3, \qquad \xi^k \in V_\xi \subset \mathbb{R}^3, \tag{2.1}$$

where $V_\xi = \{\xi^k | \xi^\alpha \in S_\xi \subset \mathbb{R}^2,\ \xi^3 \in [-1, 1]\}$ and $\boldsymbol{n}$ is the unit normal vector directed along the x_3 axis.

Differentiating both sides of equality (2.1) with respect to the variables ξ^k, we obtain the vector functions

$$\overset{0}{\boldsymbol{g}}_\alpha = \overset{0}{\boldsymbol{R}}_{,\alpha} = \overset{0}{\boldsymbol{r}}_{,\alpha}, \qquad \boldsymbol{g}_3 = \overset{0}{\boldsymbol{R}}_{,3} = h\boldsymbol{n} \qquad \Big(\overset{0}{\boldsymbol{R}}_{,\alpha} = \frac{\partial\overset{0}{\boldsymbol{R}}}{\partial\xi^\alpha}\Big), \tag{2.2}$$

which form the covariant local basis of the coordinate system ξ^k in the undeformed state.

It follows from (2.2) that the vectors $\overset{0}{\boldsymbol{g}}_\alpha$ depend only on the coordinates ξ^α and that the vector $\overset{0}{\boldsymbol{g}}_3$ is independent of ξ^k.

Since $\xi^3 \in [-1,1]$, the unknown functions $\boldsymbol{u}$ and $\hat{\boldsymbol{t}}^i$ can be expanded in series in terms of Legendre polynomials

$$\boldsymbol{u} = \sum_{k=0}^{\infty} [\boldsymbol{u}]^k P_k, \qquad \hat{\boldsymbol{t}}^i = \sum_{k=0}^{\infty} [\hat{\boldsymbol{t}}^i]^k P_k. \tag{2.3}$$

Here $P_k(\xi^3)$ are the orthogonal Legendre polynomials and $[\boldsymbol{u}]^k$ and $[\hat{\boldsymbol{t}}^i]^k$ are the expansion coefficients which depend on the coordinates $\{\xi^\alpha\} \in S_\xi \subset \mathbb{R}^2$:

$$[\boldsymbol{u}]^k = \frac{1+2k}{2} \int_{-1}^{1} \boldsymbol{u} P_k \, d\xi^3, \qquad [\hat{\boldsymbol{t}}^i]^k = \frac{1+2k}{2} \int_{-1}^{1} \hat{\boldsymbol{t}}^i P_k \, d\xi^3. \tag{2.4}$$

The integrands in (2.4) include the quantities $\hat{\boldsymbol{t}}^i$ which can be written by means of formulas (1.7) and (1.4) in the form

$$\hat{\boldsymbol{t}}^i = J\sigma^{ij} \boldsymbol{g}_j = \overset{0}{J} \tau^{ij} (\overset{0}{\boldsymbol{g}}_i + \boldsymbol{u}_{,i}). \tag{2.5}$$

3. Approximation of Stresses. We write the equations of equilibrium (1.1) in the equivalent form

$$\overset{0}{\boldsymbol{g}}{}^j \cdot (\hat{\boldsymbol{t}}^i_{,i} + \boldsymbol{f}) = 0 \qquad (j = 1,\ 2,\ 3). \tag{3.1}$$

In the undeformed state, the components of the contravariant basis $\overset{0}{\boldsymbol{g}}{}^j$ do not depend on the ξ^3 coordinate. Therefore, expanding Eqs. (3.1) in series in terms of Legendre polynomials, we obtain the system

$$\begin{gathered} \overset{0}{\boldsymbol{g}}{}^\alpha \cdot ([\hat{\boldsymbol{t}}^\alpha]^k_{,\alpha} + [\hat{\boldsymbol{t}}^3_{,3}]^k + [\hat{\boldsymbol{f}}]^k) = 0 \qquad (k = \overline{0, N+1}), \\ \overset{0}{\boldsymbol{g}}{}^3 \cdot ([\hat{\boldsymbol{t}}^\alpha]^k_{,\alpha} + [\hat{\boldsymbol{t}}^3_{,3}]^k + [\hat{\boldsymbol{f}}]^k) = 0 \qquad (k = \overline{0, N}), \end{gathered} \tag{3.2}$$

where $N \geqslant 0$ is an arbitrary number. The number of terms in expansions (3.2) is chosen in the same manner as in the linear case considered in [2, 3]. For each k, we multiply Eqs. (3.2) by P_k and summarize. As a result, we obtain

$$\overset{0}{\boldsymbol{g}}{}^\alpha \cdot (\hat{\boldsymbol{T}}'^i_{,i} + \hat{\boldsymbol{F}}) = 0, \qquad \overset{0}{\boldsymbol{g}}{}^3 \cdot (\hat{\boldsymbol{T}}''^i_{,i} + \hat{\boldsymbol{F}}) = 0. \tag{3.3}$$

Here the quantities $\hat{\boldsymbol{T}}'^i$, $\hat{\boldsymbol{T}}''^i$, and $\hat{\boldsymbol{F}}$ correspond to the truncated series

$$\begin{gathered} \hat{\boldsymbol{T}}'^\alpha = \sum_{k=0}^{N+1} [\hat{\boldsymbol{t}}^\alpha]^k P_k, \qquad \hat{\boldsymbol{T}}''^\alpha = \sum_{k=0}^{N} [\hat{\boldsymbol{t}}^\alpha]^k P_k, \\ \hat{\boldsymbol{T}}'^3 = \hat{\boldsymbol{T}}''^3 = \overset{0}{\boldsymbol{g}}_\alpha \sum_{k=0}^{N+2} ([\hat{\boldsymbol{t}}^3]^k \cdot \overset{0}{\boldsymbol{g}}{}^\alpha) P_k + \overset{0}{\boldsymbol{g}}_3 \sum_{k=0}^{N+1} ([\hat{\boldsymbol{t}}^3]^k \cdot \overset{0}{\boldsymbol{g}}{}^3) P_k, \\ \hat{\boldsymbol{F}} = \overset{0}{\boldsymbol{g}}_\alpha \sum_{k=0}^{N+1} ([\hat{\boldsymbol{f}}]^k \cdot \overset{0}{\boldsymbol{g}}{}^\alpha) P_k + \overset{0}{\boldsymbol{g}}_3 \sum_{k=0}^{N} ([\hat{\boldsymbol{f}}]^k \cdot \overset{0}{\boldsymbol{g}}{}^3) P_k. \end{gathered} \tag{3.4}$$

Thus, two approximations $\hat{\boldsymbol{T}}'^\alpha$ and $\hat{\boldsymbol{T}}''^\alpha$, which differ only by a number of terms retained in the series, correspond to the same quantities $\hat{\boldsymbol{t}}^\alpha$ in (3.3).

4. Approximation of Strains and Displacements. Let the displacement-vector variations $\delta\boldsymbol{u}$ vanish at the boundary $\overset{0}{S}_u$. For simplicity, we restrict ourselves to the case where $\hat{\boldsymbol{F}} = 0$.

From Eqs. (3.3) follows the equality

$$\int_{V_\xi} [(\overset{0}{\boldsymbol{g}}{}^\alpha \cdot \hat{\boldsymbol{T}}'^i_{,i})(\overset{0}{\boldsymbol{g}}_\alpha \cdot \delta\boldsymbol{u}) + (\overset{0}{\boldsymbol{g}}{}^3 \cdot \hat{\boldsymbol{T}}''^i_{,i})(\overset{0}{\boldsymbol{g}}_3 \cdot \delta\boldsymbol{u})] \, dV_\xi = 0 \qquad (dV_\xi = d\xi^1 \, d\xi^2 \, d\xi^3). \tag{4.1}$$

Integrating (4.1) by parts, we obtain

$$\int_{V_\xi} \{[(\overset{0}{\boldsymbol{g}}{}^\alpha \cdot \hat{\boldsymbol{T}}'^i)(\overset{0}{\boldsymbol{g}}_\alpha \cdot \delta\boldsymbol{u})]_{,i} + [(\overset{0}{\boldsymbol{g}}{}^3 \cdot \hat{\boldsymbol{T}}''^i_{,i})(\overset{0}{\boldsymbol{g}}_3 \cdot \delta\boldsymbol{u})]_{,i}\} \, dV_\xi = \int_{V_\xi} \{\hat{\boldsymbol{T}}'^i \cdot [\overset{0}{\boldsymbol{g}}{}^\alpha(\overset{0}{\boldsymbol{g}}_\alpha \cdot \delta\boldsymbol{u})]_{,i} + \hat{\boldsymbol{T}}''^i \cdot [\overset{0}{\boldsymbol{g}}{}^3(\overset{0}{\boldsymbol{g}}_3 \cdot \delta\boldsymbol{u})]_{,i}\} \, dV_\xi. \tag{4.2}$$

Using the properties of Legendre polynomials, we write the right side of (4.2) in the form

$$\int\limits_{V_\xi}\{\hat{\boldsymbol{T}}'^i\cdot[\overset{0}{\boldsymbol{g}}{}^\alpha(\overset{0}{\boldsymbol{g}}_\alpha\cdot\delta\boldsymbol{u})]_{,i}+\hat{\boldsymbol{T}}''^i\cdot[\overset{0}{\boldsymbol{g}}{}^3(\overset{0}{\boldsymbol{g}}_3\cdot\delta\boldsymbol{u})]_{,i}\}\,dV_\xi=\int\limits_{V_\xi}\{\hat{\boldsymbol{T}}'^\alpha\cdot[\overset{0}{\boldsymbol{g}}{}^\alpha(\overset{0}{\boldsymbol{g}}_\alpha\cdot\delta\boldsymbol{u})]_{,\alpha}+\hat{\boldsymbol{T}}''^\alpha\cdot[\overset{0}{\boldsymbol{g}}{}^3(\overset{0}{\boldsymbol{g}}_3\cdot\delta\boldsymbol{u})]_{,\alpha}+\hat{\boldsymbol{T}}^3\cdot\delta\boldsymbol{u}_{,3}\}\,dV_\xi$$

$$=\int\limits_{V_\xi}\left\{\hat{\boldsymbol{t}}^\alpha\cdot\left[\overset{0}{\boldsymbol{g}}{}^\alpha\left(\overset{0}{\boldsymbol{g}}_\alpha\cdot\sum_{k=0}^{N+1}[\delta\boldsymbol{u}]^kP_k\right)\right]_{,\alpha}+\hat{\boldsymbol{t}}^\alpha\cdot\left[\overset{0}{\boldsymbol{g}}{}^3\left(\overset{0}{\boldsymbol{g}}_3\cdot\sum_{k=0}^{N}[\delta\boldsymbol{u}]^kP_k\right)\right]_{,\alpha}+\hat{\boldsymbol{t}}^3\cdot\delta\boldsymbol{U}''_{,3}\right\}dV_\xi$$

$$=\int\limits_{V_\xi}\{\hat{\boldsymbol{t}}^\alpha\cdot\delta\boldsymbol{U}'_{,\alpha}+\hat{\boldsymbol{t}}^3\cdot\delta\boldsymbol{U}''_{,3}\}dV_\xi. \tag{4.3}$$

Here

$$\delta\boldsymbol{U}'=\overset{0}{\boldsymbol{g}}{}^\alpha\left(\overset{0}{\boldsymbol{g}}_\alpha\cdot\sum_{k=0}^{N+1}[\delta\boldsymbol{u}]^kP_k\right)+\overset{0}{\boldsymbol{g}}{}^3\left(\overset{0}{\boldsymbol{g}}_3\cdot\sum_{k=0}^{N}[\delta\boldsymbol{u}]^kP_k\right),\quad \delta\boldsymbol{U}''=\overset{0}{\boldsymbol{g}}{}^\alpha\left(\overset{0}{\boldsymbol{g}}_\alpha\cdot\sum_{k=0}^{N+3}[\delta\boldsymbol{u}]^kP_k\right)+\overset{0}{\boldsymbol{g}}{}^3\left(\overset{0}{\boldsymbol{g}}_3\cdot\sum_{k=0}^{N+2}[\delta\boldsymbol{u}]^kP_k\right) \tag{4.4}$$

Substituting expressions (2.5) for $\hat{\boldsymbol{t}}^i$ into (4.3) and using the symmetry of the stress tensor τ^{ij}, we perform the transformation

$$\int\limits_{V_\xi}(\hat{\boldsymbol{t}}^\alpha\cdot\delta\boldsymbol{U}'_{,\alpha}+\hat{\boldsymbol{t}}^3\cdot\delta\boldsymbol{U}''_{,3})\,dV_\xi=\int\limits_{\overset{0}{V}}(\tau^{\alpha k}\boldsymbol{g}_k\cdot\delta\boldsymbol{U}'_{,\alpha}+\tau^{3k}\boldsymbol{g}_k\cdot\delta\boldsymbol{U}''_{,3})\,d\overset{0}{V}$$

$$=\int\limits_{\overset{0}{V}}[\tau^{\alpha\beta}0.5\,(\boldsymbol{g}_\beta\cdot\delta\boldsymbol{U}'_{,\alpha}+\boldsymbol{g}_\alpha\cdot\delta\boldsymbol{U}'_{,\beta})+\tau^{3\alpha}(\boldsymbol{g}_\alpha\cdot\delta\boldsymbol{U}''_{,3}+\boldsymbol{g}_3\cdot\delta\boldsymbol{U}'_{,\alpha})+\tau^{33}(\boldsymbol{g}_3\cdot\delta\boldsymbol{U}''_{,3})]\,d\overset{0}{V},$$

$$d\overset{0}{V}=\overset{0}{J}\;d\xi^1\,d\xi^2\,d\xi^3.$$

Denoting the parentheses expressions by δE_{ij}, we obtain

$$2\delta E_{\alpha\beta}=\boldsymbol{g}_\beta\cdot\delta\boldsymbol{U}'_{,\alpha}+\boldsymbol{g}_\alpha\cdot\delta\boldsymbol{U}'_{,\beta},\quad 2\delta E_{3\alpha}=\boldsymbol{g}_\alpha\cdot\delta\boldsymbol{U}''_{,3}+\boldsymbol{g}_3\cdot\delta\boldsymbol{U}'_{,\alpha},\quad \delta E_{33}=\boldsymbol{g}_3\cdot\delta\boldsymbol{U}''_{,3}. \tag{4.5}$$

Finally, we have

$$\int\limits_{V_\xi}\{\hat{\boldsymbol{T}}'^i\cdot[\overset{0}{\boldsymbol{g}}{}^\alpha(\overset{0}{\boldsymbol{g}}_\alpha\cdot\delta\boldsymbol{u})]_{,i}+\hat{\boldsymbol{T}}''^i\cdot[\overset{0}{\boldsymbol{g}}{}^3(\overset{0}{\boldsymbol{g}}_3\cdot\delta\boldsymbol{u})]_{,i}\}\,dV_\xi=\int\limits_{\overset{0}{V}}\tau^{ij}\delta E_{ij}\,d\overset{0}{V}. \tag{4.6}$$

The quantities δE_{ij} in (4.5) approximate the variations $\delta\varepsilon_{ij}$ by truncated series in terms of Legendre polynomials [see (1.5)]. The vectors $\delta\boldsymbol{U}'$ and $\delta\boldsymbol{U}''$ [see (4.4)] are the approximations of the displacement-vector variations $\delta\boldsymbol{u}$ and they are used to calculate the derivatives with respect to the ξ^α coordinates and the ξ^3 coordinate, respectively. Bearing this in mind, we introduce the following change in expressions (1.4) for the covariant basis vectors of a deformed state $\boldsymbol{g}_i$:

$$\boldsymbol{G}_\alpha=\overset{0}{\boldsymbol{g}}_\alpha+\boldsymbol{U}'_{,\alpha},\qquad \boldsymbol{G}_3=\overset{0}{\boldsymbol{g}}_3+\boldsymbol{U}''_{,3}. \tag{4.7}$$

Substituting approximations (4.7) into expressions (4.5) in place of the covariant basis vectors of a deformed state $\boldsymbol{g}_i$, we obtain

$$2E_{\alpha\beta}=\overset{0}{\boldsymbol{g}}_\beta\cdot\boldsymbol{U}'_{,\alpha}+\overset{0}{\boldsymbol{g}}_\alpha\cdot\boldsymbol{U}'_{,\beta}+\boldsymbol{U}'_{,\alpha}\cdot\boldsymbol{U}'_{,\beta},\qquad 2E_{3\alpha}=\overset{0}{\boldsymbol{g}}_\alpha\cdot\boldsymbol{U}''_{,3}+\overset{0}{\boldsymbol{g}}_3\cdot\boldsymbol{U}'_{,\alpha}+\boldsymbol{U}'_{,\alpha}\cdot\boldsymbol{U}''_{,3},$$
$$E_{33}=\overset{0}{\boldsymbol{g}}_3\cdot\boldsymbol{U}''_{,3}+0.5\boldsymbol{U}''_{,3}\cdot\boldsymbol{U}''_{,3}. \tag{4.8}$$

Here the vectors $\boldsymbol{U}'$ and $\boldsymbol{U}''$ are written in the form of truncated series similar to those in (4.4):

$$\boldsymbol{U}'=\overset{0}{\boldsymbol{g}}{}^\alpha\left(\overset{0}{\boldsymbol{g}}_\alpha\cdot\sum_{k=0}^{N+1}[\boldsymbol{u}]^kP_k\right)+\overset{0}{\boldsymbol{g}}{}^3\left(\overset{0}{\boldsymbol{g}}_3\cdot\sum_{k=0}^{N}[\boldsymbol{u}]^kP_k\right),$$
$$\boldsymbol{U}''=\overset{0}{\boldsymbol{g}}{}^\alpha\left(\overset{0}{\boldsymbol{g}}_\alpha\cdot\sum_{k=0}^{N+3}[\boldsymbol{u}]^kP_k\right)+\overset{0}{\boldsymbol{g}}{}^3\left(\overset{0}{\boldsymbol{g}}_3\cdot\sum_{k=0}^{N+2}[\boldsymbol{u}]^kP_k\right). \tag{4.9}$$

Expressions (4.8) are the approximations E_{ij} of the Green–Lagrange strain tensor ε_{ij} (1.3).

5. Approximation of the Boundary Conditions. We denote the left side of equality (4.2) by L. Integration yields

$$L = \int\limits_{\Sigma_\xi} [(\overset{0}{\boldsymbol{g}}{}^{\alpha} \cdot \hat{\boldsymbol{T}}'^{i})(\overset{0}{\boldsymbol{g}}_{\alpha} \cdot \delta\boldsymbol{u}) + (\overset{0}{\boldsymbol{g}}{}^{3} \cdot \hat{\boldsymbol{T}}''^{i}_{,i})(\overset{0}{\boldsymbol{g}}_{3} \cdot \delta\boldsymbol{u})]\, d\xi^2\, d\xi^3 + \int\limits_{\Sigma_\xi} [(\overset{0}{\boldsymbol{g}}{}^{\alpha} \cdot \hat{\boldsymbol{T}}'^{i})(\overset{0}{\boldsymbol{g}}_{\alpha} \cdot \delta\boldsymbol{u})$$

$$+ (\overset{0}{\boldsymbol{g}}{}^{3} \cdot \hat{\boldsymbol{T}}''^{i}_{,i})(\overset{0}{\boldsymbol{g}}_{3} \cdot \delta\boldsymbol{u})]\, d\xi^1\, d\xi^3 + \int\limits_{S^+_\xi} \hat{\boldsymbol{T}}^3 \cdot \delta\boldsymbol{u}\, d\xi^1\, d\xi^2 - \int\limits_{S^-_\xi} \hat{\boldsymbol{T}}^3 \cdot \delta\boldsymbol{u}\, d\xi^1\, d\xi^2. \tag{5.1}$$

Here $\hat{\boldsymbol{T}}^3 = \hat{\boldsymbol{T}}'^3 = \hat{\boldsymbol{T}}''^3$.

We calculate the sum of the first two integrals in (5.1). To this end, using the orthogonality property of Legendre polynomials, we replace the vector $\delta\boldsymbol{u}$ by the corresponding truncated series for $\delta\boldsymbol{U}'$.

In the initial configuration, the equalities

$$d\xi^1\, d\xi^3 = \overset{0}{\nu}_2\, d\overset{0}{\Sigma}/\overset{0}{J}, \qquad d\xi^2\, d\xi^3 = \overset{0}{\nu}_1\, d\,\overset{0}{\Sigma}/\overset{0}{J} \tag{5.2}$$

are valid at the edge surface $\overset{0}{\Sigma}$ to which the coordinate line ξ^3 corresponds. Here $\overset{0}{\nu}_\alpha = \overset{0}{\boldsymbol{\nu}} \cdot \overset{0}{\boldsymbol{g}}_\alpha$ ($\overset{0}{\boldsymbol{\nu}}$ is the outward normal vector to the edge surface $\overset{0}{\Sigma}$).

Inserting (5.2) into (5.1), we obtain the sum of the first two integrals

$$\int\limits_{\overset{0}{\Sigma}} \frac{\hat{\boldsymbol{T}}^\alpha \cdot \delta\boldsymbol{U}'}{\overset{0}{J}}\, \overset{0}{\nu}_\alpha\, d\overset{0}{\Sigma}, \qquad \hat{\boldsymbol{T}}^\alpha = \overset{0}{\boldsymbol{g}}_\gamma\, (\hat{\boldsymbol{T}}'^\alpha \cdot \overset{0}{\boldsymbol{g}}{}^\gamma) + \overset{0}{\boldsymbol{g}}_3\, (\hat{\boldsymbol{T}}''^\alpha \cdot \overset{0}{\boldsymbol{g}}{}^3). \tag{5.3}$$

In the last two integrals in (5.1), which refer to the face surfaces $\overset{0}{S}{}^+$ and $\overset{0}{S}{}^-$, we transform the product $d\xi^1\, d\xi^2$, namely, $d\xi^1\, d\xi^2 = d\overset{0}{S}{}^+/\overset{0}{J} = -d\overset{0}{S}{}^-/\overset{0}{J}$.

We write equality (5.1) in the form

$$L = \int\limits_{\overset{0}{\Sigma}} \frac{\hat{\boldsymbol{T}}^\alpha \cdot \delta\boldsymbol{U}'}{\overset{0}{J}}\, \overset{0}{\nu}_\alpha\, d\overset{0}{\Sigma} + \int\limits_{\overset{0}{S}{}^+} \frac{\hat{\boldsymbol{T}}^3 \cdot \delta\boldsymbol{u}}{\overset{0}{J}}\, d\overset{0}{S}{}^+ + \int\limits_{\overset{0}{S}{}^-} \frac{\hat{\boldsymbol{T}}^3 \cdot \delta\boldsymbol{u}}{\overset{0}{J}}\, d\overset{0}{S}{}^-. \tag{5.4}$$

It follows from the first integral in (5.4) that the boundary conditions (1.8) and (1.9) at the edge surface $\overset{0}{\Sigma}$ are approximated by the truncated series

$$\boldsymbol{U}'\Big|_{\overset{0}{\Sigma}_u} = \boldsymbol{U}'_*, \quad \delta\boldsymbol{U}'\Big|_{\overset{0}{\Sigma}_u} = 0, \quad \frac{\hat{\boldsymbol{T}}^\alpha \nu^0_\alpha}{\overset{0}{J}}\Big|_{\overset{0}{\Sigma}_\sigma} = \boldsymbol{P}'_* \quad (\overset{0}{\Sigma}_u \cup \overset{0}{\Sigma}_\sigma = \overset{0}{\Sigma}). \tag{5.5}$$

Here the vectors $\boldsymbol{U}'_*$ and $\boldsymbol{P}'_*$ are written in the form of truncated series similar to the truncated series (4.9) and (5.3):

$$\boldsymbol{U}'_* = \overset{0}{\boldsymbol{g}}{}^\alpha \left(\overset{0}{\boldsymbol{g}}_\alpha \cdot \sum_{k=0}^{N+1} [\boldsymbol{u}_*]^k P_k \right) + \overset{0}{\boldsymbol{g}}{}^3 \left(\overset{0}{\boldsymbol{g}}_3 \cdot \sum_{k=0}^{N} [\boldsymbol{u}_*]^k P_k \right),$$

$$\boldsymbol{P}'_* = \overset{0}{\boldsymbol{g}}{}^\alpha \left(\overset{0}{\boldsymbol{g}}_\alpha \cdot \sum_{k=0}^{N+1} [\boldsymbol{p}_*]^k P_k \right) + \overset{0}{\boldsymbol{g}}{}^3 \left(\overset{0}{\boldsymbol{g}}_3 \cdot \sum_{k=0}^{N} [\boldsymbol{p}_*]^k P_k \right).$$

We now consider the face surfaces $\overset{0}{S}{}^+$ and $\overset{0}{S}{}^-$. In the last two integrals on the right side of (5.4), the vectors $\delta\boldsymbol{U}''$ and $\hat{\boldsymbol{T}}^3/\overset{0}{J}$ are used as variations of the displacement vector and the surface load, respectively. Therefore, the boundary conditions

$$\begin{gathered} \boldsymbol{U}''\Big|_{\overset{0}{S}{}^+_u} = \boldsymbol{u}_*, \qquad \delta\boldsymbol{U}''\Big|_{\overset{0}{S}{}^+_u} = 0, \qquad \boldsymbol{U}''\Big|_{\overset{0}{S}{}^-_u} = \boldsymbol{u}_*, \qquad \delta\boldsymbol{U}''\Big|_{\overset{0}{S}{}^-_u} = 0, \\ \frac{\hat{\boldsymbol{T}}^3}{\overset{0}{J}}\Big|_{\overset{0}{S}{}^+_\sigma} = \boldsymbol{p}_*, \qquad \frac{\hat{\boldsymbol{T}}^3}{\overset{0}{J}}\Big|_{\overset{0}{S}{}^-_\sigma} = \boldsymbol{p}_* \end{gathered} \tag{5.6}$$

are formulated at the surfaces $\overset{0}{S}{}_{\sigma}^{+}$ and $\overset{0}{S}{}_{\sigma}^{-}$. With allowance for (5.5) and (5.6), we write equality (5.4) in the final form

$$L = \int\limits_{\overset{0}{\Sigma}_{\sigma}} \boldsymbol{p}'_{*} \cdot \delta \boldsymbol{U}' \, d\overset{0}{\sigma} + \int\limits_{\overset{0}{S}{}_{\sigma}^{+}} \boldsymbol{p}_{*} \cdot \delta \boldsymbol{U}'' \, d\overset{0}{S}{}^{+} + \int\limits_{\overset{0}{S}{}_{\sigma}^{-}} \boldsymbol{p}_{*} \cdot \delta \boldsymbol{U}'' \, d\overset{0}{S}{}^{-}. \tag{5.7}$$

Thus, equality (4.2) is reduced to the form

$$\int\limits_{\overset{0}{V}} \tau^{ij} \delta E_{ij} \, d\overset{0}{V} = \int\limits_{\overset{0}{\Sigma}_{\sigma}} \boldsymbol{p}'_{*} \cdot \delta \boldsymbol{U}' \, d\overset{0}{\sigma} + \int\limits_{\overset{0}{S}{}_{\sigma}^{+}} \boldsymbol{p}_{*} \cdot \delta \boldsymbol{U}'' \, d\overset{0}{S}{}^{+} + \int\limits_{\overset{0}{S}{}_{\sigma}^{-}} \boldsymbol{p}_{*} \cdot \delta \boldsymbol{U}'' \, d\overset{0}{S}{}^{-}. \tag{5.8}$$

For the approximations introduced above, equality (5.8) is an analog of the virtual-work principle (1.10). This variational principle yields the equations of equilibrium (3.3) and the boundary conditions for stresses (5.5) and (5.6).

6. Approximation of Hooke's Law. Hooke's law (1.6) is approximated by

$$\tau^{ij} = \overset{0}{C}{}^{ijks} E_{ks}, \tag{6.1}$$

where E_{ks} are the approximations (4.8) of the Green–Lagrange strain tensor. In accordance with (2.3)–(2.5), the coefficients of series expansions of the quantities $\hat{\boldsymbol{t}}^{i}$ in terms of Legendre polynomials have the form

$$[\hat{\boldsymbol{t}}^{i}]^{k} = \frac{1+2k}{2} \int\limits_{-1}^{1} \overset{0}{J} \overset{0}{C}{}^{ijks} E_{ks} \boldsymbol{G}_{j} P_{k} \, d\xi^{3}. \tag{6.2}$$

7. System of Nonlinear Equations of the Nth Approximation. Summing up the above results, we write a two-dimensional system of nonlinear equations of the Nth approximation which comprises

— the equations of equilibrium (3.3)

$$\overset{0}{\boldsymbol{g}}{}^{\alpha} \cdot (\hat{\boldsymbol{T}}'^{i}_{,i} + \hat{\boldsymbol{F}}) = 0, \qquad \overset{0}{\boldsymbol{g}}{}^{3} \cdot (\hat{\boldsymbol{T}}''^{i}_{,i} + \hat{\boldsymbol{F}}) = 0; \tag{7.1}$$

— the relations of Hooke's law (6.1) which are written, with allowance for (6.2), in the form of truncated series (3.4)

$$\hat{\boldsymbol{T}}'^{\alpha} = \sum_{k=0}^{N+1} P_k \frac{1+2k}{2} \int\limits_{-1}^{1} \overset{0}{J} \overset{0}{C}{}^{\alpha jmn} E_{mn} \boldsymbol{G}_j P_k \, d\xi^3, \qquad \hat{\boldsymbol{T}}''^{\alpha} = \sum_{k=0}^{N} P_k \frac{1+2k}{2} \int\limits_{-1}^{1} \overset{0}{J} \overset{0}{C}{}^{\alpha jmn} E_{mn} \boldsymbol{G}_j P_k \, d\xi^3,$$

$$\hat{\boldsymbol{T}}'^{3} = \hat{\boldsymbol{T}}''^{3} = \overset{0}{\boldsymbol{g}}_{\alpha} \sum_{k=0}^{N+2} \left(P_k \frac{1+2k}{2} \, \overset{0}{\boldsymbol{g}}{}^{\alpha} \cdot \int\limits_{-1}^{1} \overset{0}{J} \overset{0}{C}{}^{3jmn} E_{mn} \boldsymbol{G}_j P_k \, d\xi^3 \right) + \overset{0}{\boldsymbol{g}}_{3} \sum_{k=0}^{N+1} \left(P_k \frac{1+2k}{2} \, \overset{0}{\boldsymbol{g}}{}^{3} \cdot \int\limits_{-1}^{1} \overset{0}{J} \overset{0}{C}{}^{3jmn} E_{mn} \boldsymbol{G}_j P_k \, d\xi^3 \right); \tag{7.2}$$

— the boundary conditions at the face surfaces (5.6)

$$\boldsymbol{U}''\Big|_{\overset{0}{S}{}_{u}^{+}} = \boldsymbol{u}_{*}, \qquad \boldsymbol{U}''\Big|_{\overset{0}{S}{}_{u}^{-}} = \boldsymbol{u}_{*}, \qquad \frac{\hat{\boldsymbol{T}}^{3}}{\overset{0}{J}}\Big|_{\overset{0}{S}{}_{\sigma}^{+}} = \boldsymbol{p}_{*}, \qquad \frac{\hat{\boldsymbol{T}}^{3}}{\overset{0}{J}}\Big|_{\overset{0}{S}{}_{\sigma}^{-}} = \boldsymbol{p}_{*}. \tag{7.3}$$

The system of nonlinear equations (7.1)–(7.3) is supplemented by the boundary conditions at the edge surfaces (5.5)

$$\boldsymbol{U}'\Big|_{\overset{0}{\Sigma}_{u}} = \boldsymbol{U}'_{*}, \qquad \frac{\hat{\boldsymbol{T}}^{\alpha} \overset{0}{\nu}_{\alpha}}{\overset{0}{J}}\Big|_{\overset{0}{\Sigma}_{\sigma}} = \boldsymbol{P}'_{*} \quad (\overset{0}{\Sigma}_{u} \cup \overset{0}{\Sigma}_{\sigma} = \overset{0}{\Sigma}),$$

$$\hat{\boldsymbol{T}}^{\alpha} = \overset{0}{\boldsymbol{g}}_{\gamma} (\hat{\boldsymbol{T}}'^{\alpha} \cdot \overset{0}{\boldsymbol{g}}{}^{\gamma}) + \overset{0}{\boldsymbol{g}}_{3} (\hat{\boldsymbol{T}}''^{\alpha} \cdot \overset{0}{\boldsymbol{g}}{}^{3}). \tag{7.4}$$

8. Linearized System of the Nth Approximation. Let the solution of the boundary-value problem (7.1)–(7.4) be known. In addition to this state, we consider a perturbed state characterized by the perturbed displacements $\tilde{\boldsymbol{U}}'$ and $\tilde{\boldsymbol{U}}''$:

$$\tilde{\boldsymbol{U}}' = \boldsymbol{U}' + \Delta \boldsymbol{U}', \qquad \tilde{\boldsymbol{U}}'' = \boldsymbol{U}'' + \Delta \boldsymbol{U}''. \tag{8.1}$$

Here the perturbation vectors $\Delta \boldsymbol{U}'$ and $\Delta \boldsymbol{U}''$ have the form of truncated series similar to the truncated series (4.9).

Assuming that the quantities $\Delta\boldsymbol{U}'$ and $\Delta\boldsymbol{U}''$ are small, the equations governing the perturbed state can be simplified by ignoring the terms nonlinear in perturbations. Denoting the terms which contain only the linear components of the perturbations $\Delta\boldsymbol{U}'$ and $\Delta\boldsymbol{U}''$ by $\tilde{(\cdot)}$, we obtain

— relations for the vectors of the covariant basis in the perturbed state

$$\tilde{\boldsymbol{G}}_i = \boldsymbol{G}_i + \Delta\boldsymbol{G}_i, \qquad \Delta\boldsymbol{G}_\alpha = \Delta\boldsymbol{U}'_{,\alpha}, \qquad \Delta\boldsymbol{G}_3 = \Delta\boldsymbol{U}''_{,3}; \tag{8.2}$$

— relations for the components of the Green–Lagrange strain tensor

$$\tilde{E}_{ij} = E_{ij} + \Delta E_{ij},$$
$$2\Delta E_{\alpha\beta} = \boldsymbol{G}_\alpha\cdot\Delta\boldsymbol{U}'_{,\beta} + \boldsymbol{G}_\beta\cdot\Delta\boldsymbol{U}'_{,\alpha}, \qquad 2\Delta E_{\alpha 3} = \boldsymbol{G}_\alpha\cdot\Delta\boldsymbol{U}''_{,3} + \boldsymbol{G}_3\cdot\Delta\boldsymbol{U}'_{,\alpha}, \tag{8.3}$$
$$\Delta E_{33} = \boldsymbol{G}_3\cdot\Delta\boldsymbol{U}''_{,3};$$

— relations for the components of the stress tensor τ^{ij}

$$\tilde{\tau}^{ij} = \tau^{ij} + \Delta\tau^{ij}, \qquad \Delta\tau^{ij} = \overset{0}{C}{}^{ijks}\Delta E_{ks};$$

— relations for the coefficients of series (3.4) $[\hat{\boldsymbol{t}}^i]^k$

$$[\tilde{\hat{\boldsymbol{t}}}{}^i]^k = [\hat{\boldsymbol{t}}^i]^k + \Delta[\hat{\boldsymbol{t}}^i]^k; \tag{8.4}$$

$$\Delta[\hat{\boldsymbol{t}}^i]^k = \frac{1+2k}{2}\int_{-1}^{1}\overset{0}{J}(\overset{0}{C}{}^{ijmn}(\Delta E_{mn}\boldsymbol{G}_j + E_{mn}\Delta\boldsymbol{G}_j)P_k)\,d\xi^3. \tag{8.5}$$

In addition to the basis vectors $\boldsymbol{G}_j$, we introduce the biorthogonal basis $\boldsymbol{G}^i$: $\boldsymbol{G}_j\cdot\boldsymbol{G}^i = \delta^i_j$ (δ^i_j is the Kronecker symbol). This allows us to show that

$$\overset{0}{C}{}^{ijmn}(\Delta E_{mn}\boldsymbol{G}_j + E_{mn}\Delta\boldsymbol{G}_j) = \tilde{C}^{ijm\alpha}(\boldsymbol{G}_m\cdot\Delta\boldsymbol{U}'_{,\alpha})\boldsymbol{G}_j + \tilde{C}^{ijm3}(\boldsymbol{G}_m\cdot\Delta\boldsymbol{U}''_{,3})\boldsymbol{G}_j,$$

where

$$\tilde{C}^{ijmn} = \overset{0}{C}{}^{ijmn} + \tau^{in}G^{mj}, \qquad G^{mj} = \boldsymbol{G}^m\cdot\boldsymbol{G}^j, \tag{8.6}$$

and reduce expressions (8.5) to the form

$$\Delta[\hat{\boldsymbol{t}}^i]^k = \frac{1+2k}{2}\int_{-1}^{1}\overset{0}{J}(\tilde{C}^{ijmn}(\boldsymbol{G}_m\cdot\Delta\boldsymbol{G}_n)\boldsymbol{G}_j)P_k\,d\xi^3. \tag{8.7}$$

Using relations (8.1)–(8.4) and (8.7), we linearize the nonlinear system (7.1)–(7.4). As a result, we obtain a linear system which comprises

— the equations of equilibrium (3.3)

$$\overset{0}{\boldsymbol{g}}{}^\alpha\cdot(\Delta\hat{\boldsymbol{T}}'^i_{,i} + \Delta\hat{\boldsymbol{F}}) = 0, \qquad \overset{0}{\boldsymbol{g}}{}^3\cdot(\Delta\hat{\boldsymbol{T}}''^i_{,i} + \Delta\hat{\boldsymbol{F}}) = 0; \tag{8.8}$$

— Hooke's law relations (8.1) written in the form of truncated series (3.4):

$$\Delta\hat{\boldsymbol{T}}'^\alpha = \sum_{k=0}^{N+1}P_k\frac{1+2k}{2}\int_{-1}^{1}\overset{0}{J}\tilde{C}^{\alpha jmn}(\boldsymbol{G}_m\cdot\Delta\boldsymbol{G}_n)\boldsymbol{G}_jP_k\,d\xi^3, \quad \Delta\hat{\boldsymbol{T}}''^\alpha = \sum_{k=0}^{N}P_k\frac{1+2k}{2}\int_{-1}^{1}\overset{0}{J}\,\tilde{C}^{\alpha jmn}(\boldsymbol{G}_m\cdot\Delta\boldsymbol{G}_n)\boldsymbol{G}_jP_k\,d\xi^3,$$

$$\Delta\hat{\boldsymbol{T}}'^3 = \Delta\hat{\boldsymbol{T}}''^3 = \overset{0}{\boldsymbol{g}}_\alpha\sum_{k=0}^{N+2}\left(P_k\frac{1+2k}{2}\,\overset{0}{\boldsymbol{g}}{}^\alpha\cdot\int_{-1}^{1}\overset{0}{J}\tilde{C}^{3jmn}(\boldsymbol{G}_m\cdot\Delta\boldsymbol{G}_n)\boldsymbol{G}_jP_k\,d\xi^3\right) \tag{8.9}$$
$$+\overset{0}{\boldsymbol{g}}_3\sum_{k=0}^{N+1}\left(P_k\frac{1+2k}{2}\,\overset{0}{\boldsymbol{g}}{}^3\cdot\int_{-1}^{1}\overset{0}{J}\tilde{C}^{3jmn}(\boldsymbol{G}_m\cdot\Delta\boldsymbol{G}_n)\boldsymbol{G}_jP_k\,d\xi^3\right);$$

— the boundary conditions at the face surfaces (5.6)

$$\Delta\boldsymbol{U}''\Big|_{\overset{0}{S}{}^{+}_{u}} = \Delta\boldsymbol{u}_*, \qquad \Delta\boldsymbol{U}''\Big|_{\overset{0}{S}{}^{-}_{u}} = \Delta\boldsymbol{u}_*, \qquad \frac{\Delta\hat{\boldsymbol{T}}^3}{\overset{0}{J}}\Big|_{\overset{0}{S}{}^{+}_{\sigma}} = \Delta\boldsymbol{p}_*, \qquad \frac{\Delta\hat{\boldsymbol{T}}^3}{\overset{0}{J}}\Big|_{\overset{0}{S}{}^{-}_{\sigma}} = \Delta\boldsymbol{p}_*. \tag{8.10}$$

The system of linear equations (8.8)–(8.11) is supplemented by the linearized boundary conditions at the edge surfaces (5.5)

$$\Delta\boldsymbol{U}'\Big|_{\overset{0}{\Sigma}_u} = \Delta\boldsymbol{U}'_*, \qquad \frac{\Delta\hat{\boldsymbol{T}}^\alpha \nu^0_\alpha}{\overset{0}{J}}\Big|_{\overset{0}{\Sigma}_\sigma} = \Delta\boldsymbol{P}'_* \qquad (\overset{0}{\Sigma}_u \cup \overset{0}{\Sigma}_\sigma = \overset{0}{\Sigma}),$$

$$\Delta\hat{\boldsymbol{T}}^\alpha = \overset{0}{\boldsymbol{g}}_\gamma (\Delta\hat{\boldsymbol{T}}'^\alpha \cdot \overset{0}{\boldsymbol{g}}{}^\gamma) + \overset{0}{\boldsymbol{g}}_3 (\Delta\hat{\boldsymbol{T}}''^\alpha \cdot \overset{0}{\boldsymbol{g}}{}^3). \tag{8.11}$$

Let us determine the differential order of the linear system (8.8)–(8.10) using the method of [3]. The strain–displacement relations (8.3) contain the coefficients of the series $\Delta\boldsymbol{U}'$ together with the first-order partial derivatives with respect to the coordinates ξ^α. Precisely these derivatives determine the differential order of the system and they must satisfy the boundary conditions at the edge surface (8.11). Six scalar coefficients of the series $\Delta\boldsymbol{U}'' - \Delta\boldsymbol{U}'$ enter the system (8.8)–(8.11) without derivatives. We call the first and second groups of unknown coefficients the basic and supplementary unknowns, respectively. The supplementary unknowns are determined from Eqs. (8.10), which are the boundary conditions at the face surfaces. These equations constitute a system of linear algebraic equations for the supplementary unknowns whose solution allows one to express the supplementary unknowns in terms of the basic unknowns. Substitution of these expressions into (8.9) leads to relations between the vector function $\Delta\hat{\boldsymbol{T}}'^\alpha$, $\Delta\hat{\boldsymbol{T}}''^\alpha$, and $\Delta\hat{\boldsymbol{T}}^3$ and the basic unknowns, i.e., the coefficients of the series $\Delta\boldsymbol{U}'$. These equations represent the linear forms relative to the coefficients of the series $\Delta\boldsymbol{U}'$ and their derivatives.

Substituting the expressions for $\Delta\hat{\boldsymbol{T}}'^\alpha$, $\Delta\hat{\boldsymbol{T}}''^\alpha$, and $\Delta\hat{\boldsymbol{T}}^3$ into the equations of equilibrium (8.8), we obtain a system consisting of $2(N+2)+1$ second-order partial scalar equations. Thus, we have a system of the order $2n$ for determination of n functions when

$$n = 2(N+2)+1. \tag{8.12}$$

The differential order of the linearized system of the Nth approximation is independent of the form of boundary conditions at the face surfaces which can be specified in terms of stresses or displacements.

The first approximation corresponds to $N = 0$. In this case, it follows from (8.12) that $n = 5$, i.e., the basic unknowns are five in number (three displacements of the middle surface and two rotations). The corresponding differential order of system (8.8)–(8.10) is equal to ten.

Using the results obtained in [3], one can obtain a system of linear equations of the Nth approximation governing the linear elastic deformation of plates which coincides with the linearized equations (8.8)–(8.10). However, there is a difference in the determination of the matrix $\tilde{C}^{ijmn}$. First, only one symmetry condition $\tilde{C}^{ijmn} = \tilde{C}^{mnij}$ is satisfied as is seen from (8.6). Second, the properties of the matrix $\tilde{C}^{ijmn}$ are determined not only by the elastic constants but also the stresses τ^{ij}. For example, for certain values of τ^{ij}, the matrix $\tilde{C}^{ijmn}$ can be not positive definite and the question of the existence and uniqueness of the solution of the linearized boundary-value problem (8.8)–(8.11) arises.

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